On Elliptic Quantum Curves in 6d SCFTs

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in collaboration with B. Haghighat, H.-C. Kim, K. Lee, M. Sperling, X. Wang, and Y. Lü

Overview

Introduction: SW-curves and their quantizations

- Quantum field theories with 8 supercharges take a special place in study of the non-perturbative phenomena in the IR physics. In particular, the low energy physics in such theories, including one-loop perturbation and non-perturbative instanton corrections, can be determined by a holomorphic function known as the prepotential.

- In the seminal work back to 90's, Seiberg and Witten showed that the prepotential in 4d $\mathcal{N}=2$ super-Yang-Mills can be determined via an algebraic curve, nowadays called Seiberg-Witten curve,

$$\mathcal{Y}(x) + \frac{\Lambda^{2N}}{\mathcal{Y}(x)} = \mathcal{W}(x; u_i)\,,$$

  where the SW-curve of 4d $\mathcal{N}=2$ pure SU(N) is illustrated, and $\mathcal{W}(x; u_i)$ is a polynomial of degree N in x, whose coefficients depend on u_i, the vevs of Coulomb branch operators in the theory.

- On the other hand, Nekrasov and Okounkov developed a powerful method to directly compute the prepotentials $\mathcal{F}$ of the theories, via the Nekrasov instanton partition function under $\Omega_{\epsilon_{1,2}}$-background,

$$\mathcal{Z}^{\text{4d}}_{\text{inst.}}(\epsilon_{1,2};\mathfrak{q}) = \sum_{k=0}^{\infty} \mathfrak{q}^k \oint_{\widetilde{\mathcal{M}_k}(\epsilon_{1,2})} 1\,, \quad \text{and} \quad \mathcal{F} = \epsilon_1\epsilon_2 \log \mathcal{Z}^{\text{4d}}_{\text{inst.}}$$

- It can be shown that the SW-curves is exactly the saddle point eq. of the instanton integral by taking $\epsilon_{1,2} \to 0$. In this picture, the variable $\mathcal{Y}(x)$ is realized as the vev of an (surface) operator $\hat{\mathcal{Y}}$,

$$\mathcal{Y}(x) \equiv \left\langle \hat{\mathcal{Y}}(x) \right\rangle .$$

- The $\mathcal{Y}$-operator is a generating function of the chiral rings of the 4d theories. As in previous example, the so-called "i-Weyl" reflection $\mathcal{Y} \to \Lambda^{2N}\mathcal{Y}^{-1}$ generates the A_1 character of the pure SU(N) theory.

- In fact, Nekrasov and Shatashvili showed that the saddle point analysis can be carried out by taking $\epsilon_2 \to 0$, while keeping $\epsilon_1 \equiv \hbar$ (NS-limit). The saddle point eq. in this procedure now defines, instead of an algebraic curve, a difference equation,

$$\mathcal{Y}(x) + \frac{\Lambda^{2N}}{\mathcal{Y}(x+\hbar)} = \mathcal{W}(x; u_i, \hbar)\,.$$

- In the NS-limit, operator $\mathcal{Y}$ can be interpreted as inserting a codimensional two defect into the theory. More specifically,

$$\Psi(x) \equiv \lim_{\epsilon_2 \to 0} \frac{\mathcal{Z}^{\mathrm{4d/2d}}_{\mathrm{inst.}}(x)}{\mathcal{Z}^{\mathrm{4d}}_{\mathrm{inst.}}}\,, \quad \text{and} \quad \mathcal{Y}(x) = \frac{\Psi(x-\hbar)}{\Psi(x)}\,,$$

where $\mathcal{Z}^{\mathrm{6d/4d}}_{\mathrm{inst.}}(x)$ is the Nekrasov partition function in precence of the codim two defect, and x now is regarded as the mass of the defect.

- In this picture, the difference eq. can be recast as

$$\left(\hat{Y} + \Lambda^{2N}\hat{Y}^{-1}\right) \cdot \Psi(x) = \mathcal{W}(x; u_i, \hbar) \cdot \Psi(x)\,,$$

where $\hat{Y} \equiv e^{-\hbar\partial_x}$ is understood as a shift operator satisfying non-trivial commutation relation with x.

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- The function $\mathcal{W}(x; u_i, \hbar)$ now is still a polynomial in x, called the fundamental q-charactor of A_1. It can be once again generated by the q-deformed i-Weyl reflectoin of operator $\mathcal{Y}$. In general $\langle\mathcal{Y}(x)\rangle$ contain poles in x, but $\mathcal{W}(x; u_i, \hbar)$ is free of poles.

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- The above illustrative example can be generalized to generic theories: The SW-curve is quantized to a Hamiltonian operator $\hat{H}(\hat{Y}, x)$, acting on a codim two defect $\Psi(x)$, and generate a codim four defect $\mathcal{W}(x)$,

$$\mathcal{H}(\hat{Y}, x) \equiv \hat{H}(\hat{Y}, x) - \mathcal{W}(x) \,, \quad \mathcal{H}(\hat{Y}, x) \cdot \Psi(x) = 0 \,.$$

- Remarkably, the algebraic SW-curves are closely related to classical (algebraic) integrable systems. The SW-curves $\mathcal{H}(y(x), x)$ can be identified as the spectral curves of the integrable systems. Their quantum version $\mathcal{H}(\hat{Y}, x)$ can be understood as the quantization of the associated integrable systems.

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- The hierarchy of the quantization of the SW-curves are summarized:

Ω-background	SW geometry	Integrable system
$(\epsilon_1, \epsilon_2) = (0, 0)$	character	classical
$(\epsilon_1, \epsilon_2) = (\hbar, 0)$	q-character	quantum
$(\epsilon_1, \epsilon_2) \neq (0, 0)$	qq-character	double quantum

- Along this line, one can study the SW-curves in 5d and 6d SCFTs, which are realized as saddle point eqs. for instanton PFs on $\mathbb{R}^4 \times S^1$ for 5d, or $\mathbb{R}^4 \times T^2$ for 6d. In this setup, the SW-curves uplift from algebraic to trigonometric in 5d and elliptic in 6d.

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- Correspondingly, the quantized curves define (relativistic) quantum trigonometirc/elliptic integrable systems,

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Now $\Psi(x)$ is determined by a 5d/3d or 6d/4d coupled system, and $\mathcal{W}(x)$ is once again the codim four defect, the Wilson loop/surface defect in 5d/6d respectively.

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- The hierarchy of the SW-curves in 4d/5d/6d are summarized:

Gauge theory	Geometric realization	Integrable system
$\mathbb{R}^4$	IIA-theory on CY3	rational
$\mathbb{R}^4 \times S^1$	M-theory on CY3	trigonometirc
$\mathbb{R}^4 \times T^2$	F-theory on CY3	elliptic

Recipe: to establish the quantum curves in 6d SCFTs

- We focus on 6d SCFTs on $\mathbb{R}^4 \times T^2$ admitting brane constructions. Therefore the ADHM constructions on their instanton string moduli sp. described by 2d $\mathcal{N} = (0,4)$ GLSMs.

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- The 6d instanton string PFs along the worldsheet T^2 are computed via the elliptic genera of 2d GLSMs,

$$
\begin{aligned}
\mathcal{Z}^{\text{6d}}_{\text{inst.}} &= \sum_{k=0}^{\infty} \sum_{k_\alpha}^{k} \prod_{\alpha} \mathfrak{q}_\alpha^{k_\alpha} \mathcal{Z}_k^{(\alpha)} \,, \\
\text{with} \quad \mathcal{Z}_k^{(\alpha)} &= \mathrm{Tr}\left((-1)^F Q^{H_L} \bar{Q}^{H_R} e^{-2\epsilon_- J_l} e^{-2\epsilon_+ (J_r - J_l)} \prod_l e^{-m_l F_l} \prod_i e^{-a_i G_i} \right) \\
&= \oint \left[\mathrm{d}\vec{\phi}^{(\alpha)} \right] Z_{\text{vec.}}(\vec{\phi}^{(\alpha)}, \vec{a}, \epsilon_{1,2}) \cdot Z_{\text{mat.}}(\vec{\phi}^{(\alpha)}, \vec{a}, \vec{m}, \epsilon_{1,2}) \,,
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$$= \oint \left[\mathrm{d}\vec{\phi}^{(\alpha)} \right] Z_{\rm vec.}(\vec{\phi}^{(\alpha)}, \vec{a}, \epsilon_{1,2}) \cdot Z_{\rm mat.}(\vec{\phi}^{(\alpha)}, \vec{a}, \vec{m}, \epsilon_{1,2}) ,$$

- The PFs can be recast in path integral formalism, and in the NS-limit,

$$\mathcal{Z}^{\rm 6d}_{\rm inst.} = \int \mathcal{D}\rho^\alpha[\phi] \exp \frac{1}{\epsilon_2} \left(\int \mathrm{d}\phi \mathrm{d}\phi' \sum_{\alpha,\beta} \rho^\alpha G_{\alpha\beta}(\phi,\phi') \rho^\beta + \int \mathrm{d}\phi \sum_\alpha \rho^\alpha \log Q_\alpha(\phi, \vec{a}, \vec{m}) \right)$$

- For $\epsilon_2 \to 0$, the saddle point eqs.,

$$\int \mathrm{d}\phi \, G_{\alpha\beta} \cdot \rho^{\beta}[\phi] + \log Q_\alpha = 0\,, \quad \text{or} \quad Q_\alpha(x) \cdot \left(e^{\int \mathrm{d}\phi \, G_{\alpha\beta} \cdot \rho^{\beta}[\phi]} \right)(x) = 1$$

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- First, a careful analysis shows that the functionals $e^{\int \mathrm{d}\phi\, G_{\alpha\beta} \cdot \rho^{\beta}[\phi]}$ are encoded by PFs in presence of various codim two defects, $\mathcal{Y}_\alpha(x)$, that are properly introduced.

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- First, a careful analysis shows that the functionals $e^{\int \mathrm{d}\phi\, G_{\alpha\beta}\cdot\rho^\beta[\phi]}$ are encoded by PFs in presence of various codim two defects, $\mathcal{Y}_\alpha(x)$, that are properly introduced.

- Secondly, the saddle point eqs. assign the q-deformed i-Weyl reflections on $\mathcal{Y}_\alpha$, from which, one can build up various q-characters $\mathcal{W}_\alpha(x)$ (as codim four Wilson surface defect) that is free of poles in x,

$$\mathcal{W}_\alpha(x) = \sum_{g \in \mathrm{iWeyl}_q} g \cdot \mathcal{Y}_\alpha(x)\,.$$

It gives the quantum Seiberg-Witten curves of the 6d SCFTs.

Ingredient 1: codimension two defects

- There are various $1/2$ BPS codim two defects. We focus on the defects introduced via higgsing meson operators in 6d, or baryons in 5d dual perspective.

- For a higgsible 6d SCFT $\mathcal{T}_n$, one can assign vevs to mesons $M = Q\tilde{Q}$,

$$\langle M \rangle = \mathrm{const.}\,,$$

  The vev triggers a RG flow, along which part of the gauge multiplets acquire masses. In the end one gets new SCFT $\mathcal{T}_m$ with lower rank m.

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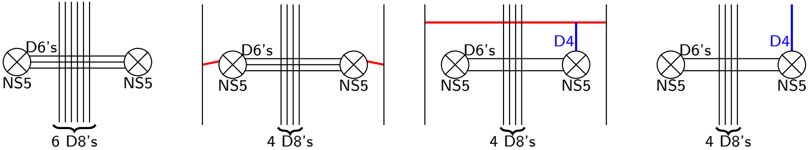
- Now we turn on a *spacetime dependent* vev to M, ($s = 1$ in this talk)

$$\langle M \rangle = z^s\,,$$

Such vev introduces a "vortex configuration" located at the z-plane, meanwhile triggers a RG flow. Now we thus end up with the SCFT $\mathcal{T}_m$ in presence of the codim two defect via the immobilized "vortex".

- An illustrative cartoon for 6d SCFT of SU(3) + 6F to SU(2) + 4F (with defect)



- The additional D4 brane gives rise to extra string modes in the 2d worldsheet GLSMs,

$$\mathcal{Z}_\alpha^{6\mathrm{d}/4\mathrm{d}}(x) = \oint \left[\mathrm{d}\vec{\phi}\right] Z_{\mathrm{vec.}} \cdot Z_{\mathrm{mat.}} \cdot Z_\alpha^{4d}(x)\,,$$

$$\text{with} \quad Z_\alpha^{4\mathrm{d}}(x) \sim \prod_{i=1}^{k} \frac{\theta_1(\phi_i + x + \epsilon_+)}{\theta_1(\phi_i + x + \epsilon_-)}\,,$$

from which, we can specify the functional $e^{\int \mathrm{d}\phi\, G_{\alpha\beta}\cdot\rho^\beta[\phi]}$ in terms of $\mathcal{Y}_\alpha$, and compute $\mathcal{Y}_\alpha(x) = \lim_{\epsilon_2\to 0} \frac{\mathcal{Z}_\alpha^{6\mathrm{d}/4\mathrm{d}}(x-\hbar)}{\mathcal{Z}_\alpha^{6\mathrm{d}/4\mathrm{d}}(x)}$ by instanton orders.

Ingredient 2: codimension four defects (Wilson surfaces)

- The ADHM construction for instanton strings can be generalized to include additional charged surface defect whose quantization gives rise to the BPS Wilson surface wrapping on the torus.
- The Wilson surface defect admits brane constructions via introducing a heavy probe string along the T^2 worldsheet. In this picture, the Wilson surface can be realized by a “double higgsing” from 2 codim two defects.

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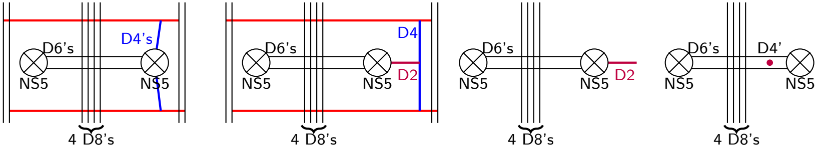
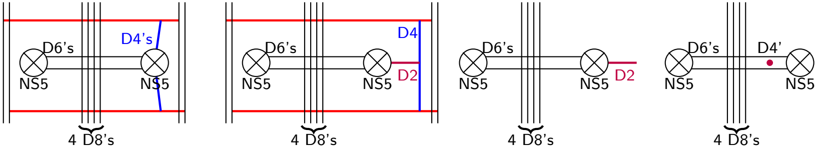
- An illustrative example of $\mathcal{S}_k$,



- The corresponding brane configuration is given by

IIA	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS5	×	×	×	×	×	×				
D6	×	×	×	×	×	×	×			
D2	×	×					×			
D4	×	×	×	×				×		
D4′	×	×						×	×	×



- The picture would be more clear, when compactifying the 6d SCFTs onto S^1 to obtain 5d KK theories which may further flow to 5d SYMs in deep IR.
- The heave probed string reduce to a heavy quark localized at the origin of $\mathbb{R}^4$. It experiences a Lorentz force proportional to gauge profiles of instanton particles, reduced from instanton strings.
- It is equivalent to insert a Wilson line defect in the instanton PFs,

$$\mathcal{Z}^{6\mathrm{d}/2\mathrm{d}}(x) = \mathcal{Z}^{5\mathrm{d}/1\mathrm{d}}_{\mathrm{KK}}(x) \sim \int \cdots \mathcal{D}\chi e^{\cdots + \int \mathrm{d}t\, \chi^\dagger (\partial_t - iA_t + \Phi - x)\chi} .$$

After integrating out the heave quark ψ, $\mathcal{Z}^{5\mathrm{d}/1\mathrm{d}}_{\mathrm{KK}}(x)$ can be understood as a generating function of Wilson loops in various Reps.,

$$\mathcal{Z}^{5\mathrm{d}/1\mathrm{d}}_{\mathrm{KK}}(x) = \sum_\alpha W_\alpha b^{[\alpha]}(x) ,$$

where $W_\alpha = \mathrm{Tr}_{R_n} P \exp \int \mathrm{d}t\, (A_t + i\Phi)$ is the Wilson loop in Reps. R_n, and $b^{[\alpha]}(x)$ are bases expanding $\mathcal{Z}^{5\mathrm{d}/1\mathrm{d}}(x)$.

- A remark on the codim four defects in 4d/5d/6d in general,

$$\mathcal{Z}^{\mathrm{4d/0d}}(x) = \sum_{\alpha} u_\alpha x^\alpha\,, \quad \text{with} \quad u_\alpha \equiv \langle \mathrm{Tr}\,\Phi^\alpha \rangle\,;$$

$$\mathcal{Z}^{\mathrm{5d/1d}}(x) = \sum_{\alpha} W_\alpha X^\alpha\,, \quad \text{with} \quad X \equiv e^{-x};$$

$$\mathcal{Z}^{\mathrm{6d/2d}}(x) = \mathcal{Z}^{\mathrm{5d/1d}}_{\mathrm{KK}}(x) = \sum_{\alpha} W_\alpha\, \theta^{[\alpha]}(x)\,,$$

where $\theta^{[\alpha]}(x)$ are bases of degree-α elliptic functions on the torus.

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where $\theta^{[\alpha]}(x)$ are bases of degree-α elliptic functions on the torus.

- The 6d PFs in presence of codim four defect, can be either computed from 6d or 5d perspectives. On the level of PFs,

$$\mathcal{Z}^{\mathrm{6d/2d}}_\alpha(x) = \oint \left[\mathrm{d}\vec{\phi}\right] Z_{\mathrm{vec.}} \cdot Z_{\mathrm{mat.}} \cdot Z^{2d}_\alpha(x) \,,$$
$$\text{with} \quad Z^{\mathrm{2d}}_\alpha(x) \sim \prod_{i=1}^{k} \frac{\theta_1(\epsilon_- \pm (\phi_i + x))}{\theta_1(-\epsilon_+ \pm (\phi_i + x))} \,,$$

from which, one can compute $\mathcal{Z}^{\mathrm{6d/2d}}(x)$ via localization by instanton orders.

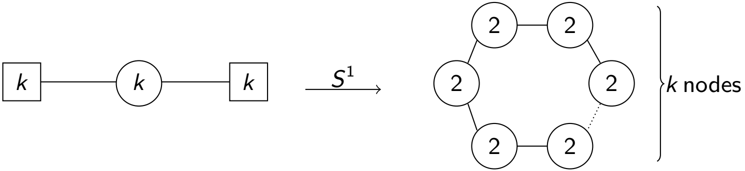
- Finally, we assemble the two ingredients, the codim two/four defects, by the recipe: In the context of quantum Seiberg-Witten curves, the claim is, *the normalized codim four defects* $\mathcal{Z}^{\mathrm{6d/2d}}(x)$*, under NS-limit, equals the q-characters from codim two defects* $\mathcal{Y}$,

$$\mathcal{Y}_\alpha(x) = \lim_{\epsilon_2 \to 0} \frac{\mathcal{Z}_\alpha^{\mathrm{6d/4d}}(x-\hbar)}{\mathcal{Z}_\alpha^{\mathrm{6d/4d}}(x)}\,, \quad \mathcal{W}_\alpha(x) = \lim_{\epsilon_2 \to 0} \frac{\mathcal{Z}_\alpha^{\mathrm{6d/2d}}(x)}{\mathcal{Z}^{\mathrm{6d}}}$$

$$\mathcal{W}_\alpha(x) = \sum_{g \in \mathrm{iWeyl}_q} g \cdot \mathcal{Y}_\alpha(x)\,.$$

Example 1: $\mathcal{S}_k$ class (single tensor) [JC, Haghighat, Kim & Sperling; 20]

- The 6d SCFTs is realized by 2 M5 branes probing $\mathbb{Z}_k$ singularity,



- Only one codim two defect can be introduced from higgsing $\mathcal{S}_{k+1}$ to $\mathcal{S}_k$, and one $\mathcal{Y}$-function is defined,

$$\Psi(x) = \lim_{\epsilon_2 \to 0} \frac{\mathcal{Z}^{\mathrm{6d/4d}}(x)}{\mathcal{Z}^{\mathrm{6d}}} \implies \mathcal{Y}(x) = \frac{\Psi(x-\hbar)}{\Psi(x)}\,,$$

- Saddle point eq. gives,

$$\mathcal{Y}(u) + \frac{Q(u)}{\mathcal{Y}(u+\hbar)} = 0\,, \quad \text{with} \quad Q(x) = \prod_{i=1}^{2k} \theta_1(x - m_i)\,,$$

- The q-deformed iWeyl reflection s is given by

$$s: \quad \mathcal{Y}(x) \mapsto \frac{Q(x)}{\mathcal{Y}(x+\hbar)},$$

and the q-character

$$\mathcal{W}(x) = \mathcal{Y}(x) + \frac{Q(x)}{\mathcal{Y}(x+\hbar)}.$$

- $\mathcal{W}(x)$ can be verified as normalized Wilson surface defect under NS-limit, specifically,

$$\mathcal{W}(x) = \sum_{n=1}^{k} W_n \cdot \theta^{[n]}(x),$$

where W_n are the q-deformed Wilson lines in fund. Rep. of each gauge node from 5d perspective.[H.-C. Kim, M. Kim, S.-S. Kim]

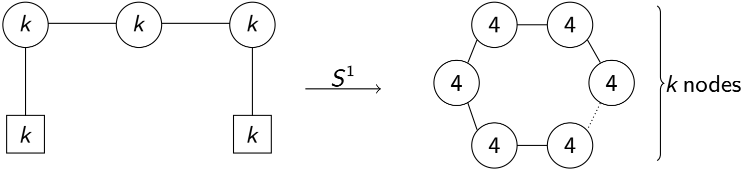
- The quantum curve,

$$\mathcal{H}(\hat{Y}, x) = \hat{Y} + Q(x)\hat{Y}^{-1} - \mathcal{W}(x),$$

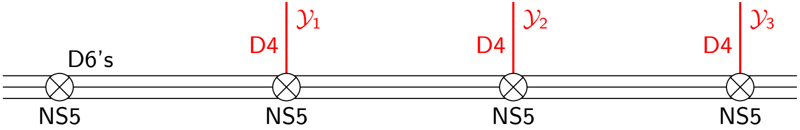
can be identified as two-body Ruijsennars-Schneider model enriched by $SU(2k)$ flavors.

Example 2: $\mathcal{S}_k$ class (multiple tensors) [In progress]

- The 6d SCFTs is realized by N M5 branes probing $\mathbb{Z}_k$ singularity, (e.g. N=4)



- We higgs the giant meson M from $\mathcal{S}_{k+1}$ to $\mathcal{S}_k$, and able to define 3 $\mathcal{Y}_\alpha$ from the brane picture,



- From saddle point eqs, we obtain 3 q-deformed iWeyl s_i reflections,

$$s_1\,:\,\begin{cases}\mathcal{Y}_1(x)\to\frac{Q_1(x)\,\mathcal{Y}_2(x+\frac{\hbar}{2})}{\mathcal{Y}_1(x+\hbar)}\\ \mathcal{Y}_i(x)\to\mathcal{Y}_i(x)\,,\quad\text{for}\quad i\neq 1\,,\end{cases}$$

$$s_2\,:\,\begin{cases}\mathcal{Y}_2(x)\to\frac{Q_2(x)\,\mathcal{Y}_1(x+\frac{\hbar}{2})\,\mathcal{Y}_3(x+\frac{\hbar}{2})}{\mathcal{Y}_2(x+\hbar)}\\ \mathcal{Y}_i(x)\to\mathcal{Y}_i(x)\,,\quad\text{for}\quad i\neq 2\,,\end{cases}$$

$$s_3\,:\,\begin{cases}\mathcal{Y}_3(x)\to\frac{Q_3(x)\,\mathcal{Y}_2(x+\frac{\hbar}{2})}{\mathcal{Y}_3(x+\hbar)}\\ \mathcal{Y}_i(x)\to\mathcal{Y}_i(x)\,,\quad\text{for}\quad i\neq 3\,,\end{cases}$$

which determine the orbits of $\mathcal{Y}_\alpha$:

$$\mathcal{Y}_1\xrightarrow{s_1}\frac{Q_1\mathcal{Y}_2}{\mathcal{Y}_1}\xrightarrow{s_2}\frac{Q_1Q_2\mathcal{Y}_3}{\mathcal{Y}_2}\xrightarrow{s_3}\frac{Q_1Q_2Q_3}{\mathcal{Y}_3}$$

$$\begin{tikzcd}
 & & \frac{Q_1Q_2\mathcal{Y}_3}{\mathcal{Y}_1} \arrow[dr, "s_3"] & & \\
\mathcal{Y}_2 \arrow[r, "s_2"] & \frac{Q_2\mathcal{Y}_1\mathcal{Y}_3}{\mathcal{Y}_2} \arrow[ur, "s_1"] \arrow[dr, "s_3"'] & & \frac{Q_1Q_2Q_3\mathcal{Y}_2}{\mathcal{Y}_1\mathcal{Y}_3} \arrow[r, "s_2"] & \frac{Q_1Q_2^2Q_3}{\mathcal{Y}_2} \\
 & & \frac{Q_2Q_3\mathcal{Y}_1}{\mathcal{Y}_3} \arrow[ur, "s_1"'] & &
\end{tikzcd}$$

$$\mathcal{Y}_3\xrightarrow{s_3}\frac{Q_3\mathcal{Y}_2}{\mathcal{Y}_3}\xrightarrow{s_2}\frac{Q_2Q_3\mathcal{Y}_1}{\mathcal{Y}_2}\xrightarrow{s_1}\frac{Q_1Q_2Q_3}{\mathcal{Y}_1}$$

- The orbits of $\mathcal{Y}_\alpha$ determines $\mathcal{W}_\alpha$,

$$
\begin{aligned}
\mathcal{W}_1(x) &= \mathcal{Y}_1(x) + \frac{Q_1(x)\,\mathcal{Y}_2(x+\frac{\hbar}{2})}{\mathcal{Y}_1(x+\hbar)} + \frac{Q_1(x)Q_2(x+\frac{\hbar}{2})\,\mathcal{Y}_3(x+\hbar)}{\mathcal{Y}_2(x+\frac{3\hbar}{2})} + \frac{Q_1(x)Q_2(x+\frac{\hbar}{2})Q_3(x+\hbar)}{\mathcal{Y}_3(x+2\hbar)} \\
\mathcal{W}_2(x) &= \mathcal{Y}_2(x) + \frac{Q_2(x)\,\mathcal{Y}_1(x+\frac{\hbar}{2})\,\mathcal{Y}_3(x+\frac{\hbar}{2})}{\mathcal{Y}_2(x+\hbar)} + \frac{Q_2(x)Q_3(x+\frac{\hbar}{2})\,\mathcal{Y}_1(x+\frac{\hbar}{2})}{\mathcal{Y}_3(x+\frac{3\hbar}{2})} + \frac{Q_1(x+\frac{\hbar}{2})Q_2(x)\,\mathcal{Y}_3(x+\frac{\hbar}{2})}{\mathcal{Y}_1(x+\frac{3\hbar}{2})} \\
&\quad + \frac{Q_1(x+\frac{\hbar}{2})Q_2(x)Q_3(x+\frac{\hbar}{2})\,\mathcal{Y}_2(x+\hbar)}{\mathcal{Y}_1(x+\frac{3\hbar}{2})y_3(x+\frac{3\hbar}{2})} + \frac{Q_1(x+\frac{\hbar}{2})Q_2(x)Q_2(x+\hbar)Q_3(x+\frac{\hbar}{2})}{\mathcal{Y}_2(x+2\hbar)}\,, \\
\mathcal{W}_3(x) &= \mathcal{Y}_3(x) + \frac{Q_3(x)\,\mathcal{Y}_2(x+\frac{\hbar}{2})}{\mathcal{Y}_3(x+\hbar)} + \frac{Q_2(x+\frac{\hbar}{2})Q_3(x)\,\mathcal{Y}_1(x+\hbar)}{\mathcal{Y}_2(x+\frac{3\hbar}{2})} + \frac{Q_1(x+\hbar)Q_2(x+\frac{\hbar}{2})Q_3(x)}{\mathcal{Y}_1(x+2\hbar)}\,.
\end{aligned}
$$

- The above q-character $\mathcal{W}_\alpha$ are expected to agree with Wilson surface defects, introduced via the brane picture,

- For $\mathcal{W}_\alpha(x)$ are free of poles, and admits expansions for certain bases,

$$\mathcal{W}_\alpha(x) = \sum_{n=1}^{k} W_{\alpha,\,n} \cdot \theta^{[n]}(x)\,,$$

where $W_{\alpha,\,n}$ should correspond to $3k$ Wilson loops in $\Lambda^{1,\,2,\,3}$-AS Reps. of the k $SU(4)$ gauge nodes from 5d perspective.

- From $\mathcal{W}_\alpha$, reconstruct the Hamiltonian associated to $\Psi_1(x)$ as

$$\mathcal{H}_1(\hat{Y}, x) = \hat{Y} - P_1(x) + P_2(x)\,\hat{Y}^{-1} - P_3(x)\,\hat{Y}^{-2} + P_4(x)\,\hat{Y}^{-3}$$

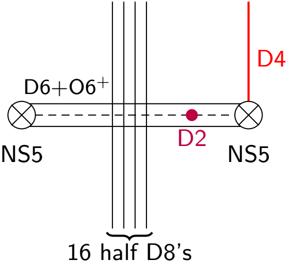
with

$$\begin{cases} P_1(x) = \mathcal{W}_1(x) \\ P_2(x) = Q_1(x)\,\mathcal{W}_2(x+\frac{\hbar}{2}) \\ P_3(x) = Q_1(x)Q_1(x+\hbar)Q_2(x+\frac{\hbar}{2})\,\mathcal{W}_3(x+\hbar) \\ P_4(x) = Q_1(x)Q_1(x+\hbar)Q_1(x+2\hbar)Q_2(x+\frac{\hbar}{2})Q_2(x+\frac{3\hbar}{2})Q_3(x+\hbar) \end{cases}$$

- $\mathcal{H}_1(\hat{Y}, x)$ can be identified as the spectral curve of the 4-body RS model enriched with $SU(2k)$ flavors.

Example 3: E-string [JC, Haghighat, Kim, Sperling & Wang; 21]

- The 6d E-string is realized by an M5 branes probing D_4 singularity, or in an NS5-D6/O6$^+$-D8 brane system in type IIA.
- The codim two/four defects are similarly to be introduced as in $\mathcal{S}_k$,



- The q-character $\mathcal{W}$ is given by

$$\mathcal{W}(x) = \mathcal{Y}(x) + \frac{Q(-x)Q(x+\hbar)}{\mathcal{Y}(x+\hbar)}, \quad \text{with} \quad Q(x) = \frac{\prod_{i=1}^{8} \theta_1(x+m_i)}{\theta_1(2x)\theta_1(2x+\hbar)},$$

- $\mathcal{W}(x)$ is verified to be the Wilson surface defect. However it contains poles in x at the 1-instanton order. In fact, we find

$$\mathcal{W}(x) = -W_1(x) + \mathcal{W}^{\mathrm{5d/1d}}$$
$$\text{with} \quad W_1(x) = \sum_{I=1}^{4} \frac{\prod_i \theta_I(m_i)}{2\eta^9 \theta_1(\hbar)} \left(\frac{\theta_I'(x-\frac{\hbar}{2})}{\theta_I(x-\frac{\hbar}{2})} - \frac{\theta_I'(x+\frac{\hbar}{2})}{\theta_I(x+\frac{\hbar}{2})} \right) ,$$

where $\mathcal{W}^{\mathrm{5d/1d}}$ is identified to the Wilson loop of the 5d SU(2) with 8 flavors, and displays a E_8 symmetry enhancement.

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where $\mathcal{W}^{\mathrm{5d/1d}}$ is identified to the Wilson loop of the 5d SU(2) with 8 flavors, and displays a E_8 symmetry enhancement.

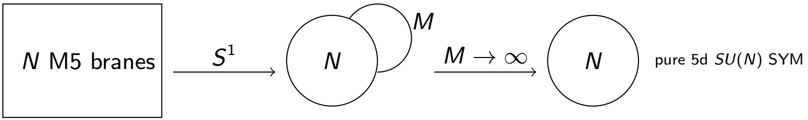
- On the other hand, $W_1(x)$ is remarkably identified to the famous 4-theta potential in van-Diejen model. The quantum curve

$$\mathcal{H}(\hat{Y}, x) = \hat{Y} + Q(-x)Q(x+\hbar)\hat{Y}^{-1} + W_1(x) - \mathcal{W}^{\mathrm{5d/1d}} = 0,$$

is thus recognized as the van-Diejen model

Example 3.5: E-string curve cascade [JC, Lü & Wang; in progress]

- It has been long time known that the relativistic Toda chain can be obtained via degeneration of n-body RS models (elliptic),



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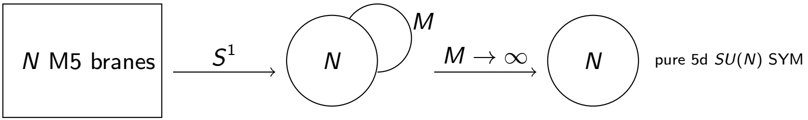

- Along this line, can we establish a series of degenerations of E-string quantum curves?

- Starting from E-string curve/van Diejen $\hat{H}(x, \{m_i\}_{i=1,\dots,8;q,\hbar})$, can take a series of (mass) limits, w.r.t. the corresponding del Pezzo geometries, to establish various degenerate curves.

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- The degeneration series are expected to quantum curves of 5d SCFTs of $SU(2)$ with $N_f = 7, 6, 5, 4$ flavors, whose quantum curves are trigonometric, and have enhanced E_8, E_7, E_6, D_5 symmetries,

van Diejen with E_8 → 1st deg. with E_8 → 2nd deg. with E_7 → 2nd deg. with E_6 → 2nd deg. with D_5

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$$\boxed{\begin{matrix}\text{van Diejen}\\ \text{with } E_8\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{1st deg.}\\ \text{with } E_8\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{2nd deg.}\\ \text{with } E_7\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{2nd deg.}\\ \text{with } E_6\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{2nd deg.}\\ \text{with } D_5\end{matrix}}$$

- Remarkably, from the integrable system perspective, it has been recently shown that the reduced (quantum) hamiltons from van Diejen also display exactly the same symmetries [Takemura; 16], [Sasaki, Takagi & Takemura; 21]. There are also many progresses to study these 6d/5d (classical/quantum) curves via brane-web/del Pezzo geometries. [Kim, Sugimoto & Yagi; 20], [Kim, Sugimoto & Yagi; 22?]; [Moriyama; 20], [Moriyama & Yamada; 21]

Example 4: 6d $SO(2N)$ on -4 curve [JC, Haghighat, Kim, Lee, Sperling & Wang; 21]

- The 6d SCFT is realized by a system of NS5-D6/O6$^-$-D8 branes
- The quantum SW-curve is given by

$$\mathcal{H}(\hat{Y}, x) = \hat{Y} + Q(-x)Q(x+\hbar)\hat{Y}^{-1} - \mathcal{W}(x)\,,$$

$$\text{with} \quad Q(x) = \theta_1(2x)\theta_1(2x+\hbar)\prod_{i=1}^{2N-8}\theta_1(x \pm m_i + \frac{\hbar}{2})$$

It should define a elliptic integrable model with Sp(4N-16) symmetry.

- The Wilson surface defect,

$$\mathcal{W}(x) = \sum_{i=0}^{N} W_i \cdot \theta_2(2x|2\tau)^i \theta_3(2x|2\tau)^{N-i}\,,$$

gives Wilson loops W_i in fund. Reps. of SU(2) nodes in the $\widehat{D}_N$ quiver from 5d perspective.

Example 4.5: D-type CM & -1-4-1 Necklace, [JC, Lü & Wang; in progress]

- The D-type minimal conformal matter 6d SCFTs are Sp(N-4) on "-1 curve", a generalization of E-string. We established their quantum curves and identified them as a type of elliplitc Garnier systems.

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$$\overset{\mathfrak{sp}(k_1)}{\mathbf{1}} \text{———} \overset{\mathfrak{so}(k_2)}{\mathbf{4}} \text{———} \overset{\mathfrak{sp}(k_3)}{\mathbf{1}} \text{———} \overset{\mathfrak{so}(k_4)}{\mathbf{4}}$$

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- One can close the "-1-4-1" necklace, and make the quiver affine. It gives the Sp-SO little string theories. Its classical curve has been studied in recent years by different approaches [Hagighat, Kim, Yan & Yau; 18], [Kim, Sugimoto & Yagi; 22?]. It is very interesting to study the quantum version of it.

Outlooks: work in progress and future

- Some of furture direction has been shown in the examples. Basically there are two basic operations:

  degenerations and gluings of various elliptic quantum curves as building blocks. Both of them will give us quantum curve cascades.

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  Quantum curves in minimal SCFTs: $\mathrm{SU}(3)$ on -3, F_4 on -5, $E_{6,7,8}$ on -6, -7, -8, -12 curves. Their quantum curves, due to no known brane constructions, are challenging, but very interesting to study.

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- Bridging elliptic integrable systems from elliptic quantum curves:

  Since the 6d SCFTs have been classified from F-theory, can one expect to have a (partial) classification of the elliptic integrable systems?

Historic stuffs

John Scott Russell and the solitary wave [1844]

"...I followed it [wave] on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height."

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- THANK YOU FOR YOUR ATTENTION!